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Non-periodic long-range order for one-dimensional pair interactions

Aernout C D van Enter†§ and Boguslaw Zegarlinski‡‖

† Institute for Theoretical Physics, Rijksuniversiteit Groningen, PO Box 800, 9747 AG Groningen, The Netherlands

‡ Department of Mathematics, Imperial College, London SW7 2BZ, UK

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Abstract. We show how non-periodic (quasiperiodic) long-range order can arise in one dimension due to long-range, translation-invariant pair interactions with quasiperiodically alternating signs. We discuss the Parisi overlap distribution between the infinitely many pure states.

1. Introduction

Since the discovery of quasicrystals [27], the description of systems which are ordered in a non-periodic manner, either quasiperiodically or even more irregularly, has been a topic of considerable interest and physical relevance, see for example [12, 11, 18, 26] and references therein. Moreover, the study of such irregular types of long-range order may be of use in understanding some conceptual aspects of the spin-glass problem.

One question of interest is to show the possibility that such a non-periodic long-range order arises for spin models with translation-invariant interactions, in which the spins can take a finite number of values. At zero temperature this has been achieved for models of spatial dimension at least 2, mostly on a lattice, with nearest-neighbour interactions, based on the theory of non-periodic tilings. In one-dimensional models, one needs infinite-range interactions [23], which, however, can decay arbitrarily quickly and can be chosen to be of finite-body type [9]. At positive temperatures there have up to now been only limited results. To obtain any type of long-range order in one dimension, the interactions are necessarily of infinite range, and, moreover, they have to decay rather slowly. In [6] it was shown that for some long-range four-body interaction, of a rather artificial form, long-range order of Thue–Morse type can occur. A related argument for pair interactions with huge gaps (that is, couplings are non-zero only for pairs of spins at exceptional distances from each other) was also indicated. Here we use similar ideas to show the possibility of *a quasiperiodic long-range order occurring for a translation-invariant pair interaction of alternating sign*, in such a way that the sign of the correlation function essentially follows the sign of the interaction. In fact, our argument is in some sense inverted, in that the sign of the sought-after interaction follows the sign of some quasiperiodic correlation function.

Long-range translation-invariant pair interactions with alternating sign are physically much more realistic than the interactions considered in [6]. One of the best-known examples

§ aenter@th.rug.nl
‖ boz@ic.ac.uk

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is the RKKY interaction, which plays an important role in the modelling of spin-glasses. For a recent review on results for long-range interactions we refer the reader to [28]. We remark that a system with non-periodic long-range order has infinitely many pure states, a property which is often suggested as being characteristic for spin-glasses. (For a further discussion about analogies between non-periodic systems and spin-glasses, we refer the reader to [5, 7, 28].) The main difference seems to be the absence of disorder; however, there have also been attempts to describe spin-glass behaviour with non-random models [4, 15, 16, 2, 3, 24].

As an aside, we observe that it is much simpler to obtain a non-periodic (helical) long-range order for continuous vector spin models (compare [25, p 163, note 26]). One can consider, for example, a rotator model with Hamiltonian

$$H = -\sum_{i,j} J(i-j)\cos(\theta_i - \theta_j - (i-j)\times\alpha) \tag{1.1}$$

for α an irrational multiple of 2π, which by a simple change of variables is equivalent to a ferromagnetic model if the $J(i-j)$ are positive. The ferromagnetic long-range order of the transformed model is equivalent to a non-periodic long-range order in the original model.

2. The result

In [28] it was conjectured that under reasonable conditions an Ising pair interaction whose Fourier transform has its maximum at an irrational point should show non-periodic long-range order. It can easily be seen, as is also noticed there, that arguments implying that the behaviour of the Fourier transform of the interaction function governs the nature of the phase transition cannot be valid in complete generality. Here we show some results which at least go in the right direction towards confirming this picture.

We consider the circle $[0, 2\pi)$ and the irrational rotation map (over an irrational angle α):

$$T_\alpha\theta = \theta + \alpha. \tag{2.1}$$

With any θ we associate the sequence $\sigma_\alpha(\theta) \in \{-1,+1\}^Z$ defined by $\sigma_{n,\alpha}(\theta) = 1$ if $\theta + n\alpha \pmod{2\pi} \in [-\frac{1}{2}\pi, +\frac{1}{2}\pi)$ and $\sigma_{n,\alpha}(\theta) = -1$ if $\theta + n\alpha \pmod{2\pi} \in [\frac{1}{2}\pi, \frac{3}{2}\pi)$. Here α is an irrational multiple of 2π.

Then the irrational rotation corresponds to the translation map on these sequences, and the rotation invariant Lebesgue measure on the circle corresponds to a translation-invariant, ergodic measure μ_α which is induced on the space $\{-1,+1\}^Z$ by the map $\theta \to \sigma_\alpha(\theta)$. It is straightforward to compute the pair correlation functions of these measures, and to see that they have a non-periodic but quasiperiodic long-range order. Indeed, let

$$\mu_\alpha(\sigma_0\sigma_n) = f_\alpha(n) \tag{2.2}$$

with

$$f_\alpha(n) = 1 - \left|\frac{((n\times\alpha)\ (\mathrm{mod}\ 2\pi))}{\pi}\right| \tag{2.3}$$

then the $f_\alpha(n)$ form a quasiperiodic sequence of pair correlation functions. (The Fourier transform of these pair correlations lives on a dense set of points in the interval $[0, 2\pi)$, given by the integer multiples of $\alpha \pmod{2\pi}$.) The relation $\mathrm{sgn}\,(f_\alpha(n)) = \mathrm{sgn}\,(\cos n\alpha)$ holds.

We will apply the Israel–Bishop–Phelps theorem [1, 13], [14, chapter V], [10, chapter 16], to conclude the existence of a similar long-range order for an appropriate Gibbs measure on the configuration-space (space of sequences) $\{-1,+1\}^Z$. For the theory of Gibbs measures for lattice models, we refer the reader to [8, 10, 14]. Applying the Israel–Bishop–Phelps theorem works in the same way in any dimension, for simplicity we thus consider here the one-dimensional case only. Let us consider in the following cones of translation-invariant interactions given by:

$$C_\alpha = \left\{ J(n) : \operatorname{sgn}(J(n)) = \operatorname{sgn}(f_\alpha(n)),\ \sum_n |J(n)| < \infty \right\} \tag{2.4}$$

such that one has formal Ising Hamiltonians of the form

$$H = -\sum_{i,j \in Z} J(i-j)\sigma_i \sigma_j. \tag{2.5}$$

The Israel–Bishop–Phelps theorem in our case reads as follows:

Theorem 2.1. For every irrational α and every $\epsilon > 0$ there exists a translation-invariant pair interaction $J_{\epsilon,\alpha} \in C_\alpha$, such that $\sum_n |J_{\epsilon,\alpha}(n)| \leqslant (\ln 2)/\epsilon$, for which there exists a translation-invariant Gibbs measure $\mu_{J_{\epsilon,\alpha}}$ which satisfies

$$\operatorname{sgn}(\mu_\alpha(\sigma_0\sigma_n))[\mu_{J_{\epsilon,\alpha}}(\sigma_0\sigma_n) - \mu_\alpha(\sigma_0\sigma_n)] \geqslant -\epsilon. \tag{2.6}$$

This means that there exists a Gibbs measure with the same kind of quasiperiodic long-range order in the two-point correlation functions as occurs in μ_α, while the sign of the interaction between spins at distance n coincides with the sign of the correlation function at distance n, at least when this correlation function is not too close to zero. For pair interactions this puts a limit on the unavoidable frustration. *The decomposition into extremal Gibbs measures of any such a $\mu_{J_{\epsilon,\alpha}}$ contains uncountably infinitely many elements* (for any extremal non-periodic Gibbs measure occurring in the decomposition its countably many translations occur with the same weight, which thus has to be zero; as countably many elements of measure zero still only add up to a measure zero contribution, one needs to have uncountably infinitely many extremal elements in the decomposition).

To get some more control over the cone of interactions, which gets closer to the Fourier transform having a maximum at α, at the cost of losing some control over the correlation functions, we can replace C_α by its subcone

$$C'_\alpha = \left\{ J(n) : \operatorname{sgn}(J(n)) = \operatorname{sgn}(f_\alpha(n)),\ \sum_n \frac{J(n)}{\cos n\alpha} < \infty \right\}. \tag{2.7}$$

This means that in the Fourier series $\sum_n J(n)\cos nx$, at the point $x = \alpha$ not only are all the terms positive, but the $J(n)$ have to be small whenever the term $\cos n\alpha$ is small.

For the corresponding Gibbs measure for some element of the subcone we obtain the inequality

$$\frac{1}{\cos n\alpha}\left[\mu_{J_{\epsilon,\alpha}}(\sigma_0\sigma_n) - \mu_\alpha(\sigma_0\sigma_n)\right] \geqslant -\epsilon \tag{2.8}$$

which still implies that non-periodic long-range order occurs.

Remark. Similarly to the case of the Fibonacci system [5], it is possible to compute the Parisi overlap distribution $p(q)$ of the μ_α. Let us recall that the overlap between two configurations σ^1 and σ^2 is given by

$$q = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1,\ldots,N} \sigma_n^1 \sigma_n^2. \tag{2.9}$$

Its distribution with respect to the product-measure $\mu_\alpha \otimes \mu_\alpha$ is the Parisi overlap distribution which plays an important role in the mean-field theory of spin-glasses [17]. In our case it is given by

$$p(q)\mathrm{d}q = \tfrac{1}{2}1_{[-1,+1]}\mathrm{d}q \tag{2.10}$$

which, as one sees, is non-trivial, continuous, and is the same for each α. This behaviour for the overlap distribution, as well as the earlier results of [5], illustrates the point emphasised in [19–21] that (also non-trivial) overlap-distributions in great generality are self-averaging. Here the self-averaging property is trivially fulfilled (there is no averaging to be done), but the overlap distribution can be computed explicitly and shown to be non-trivial.

3. Comments and conclusions

We have used the Israel–Bishop–Phelps theorem to show the occurrence of quasiperiodic long-range order at positive temperatures for translation-invariant pair interactions of (quasiperiodically) alternating signs. As usual the problem with applying the Israel–Bishop–Phelps theorem is its non-constructive character. One knows that there exists an interaction within a certain class of interactions whose Gibbs states give rise to some prescribed type of long-range order, but one does not know this interaction precisely. However, in cases where there is no reflection positivity available, and also no contour argument has been found, this is essentially the only rigorous method available to us. It should be helpful for further progress in this domain to see that a phase transition with non-periodic long-range order (with infinitely many pure Gibbs phases) can occur for a certain type of interaction. Compared with the earlier results of [6], the class of interactions we consider contains physically much more realistic potentials. Moreover, our results link for the first time the Israel–Bishop–Phelps theorem with classes of interactions whose Fourier transforms have certain prescribed properties. Attempts to read off the behaviour of the long-range correlation functions and the phase transition from the properties of the Fourier transform of an interaction are of course widespread; they form the foundation of spin-wave theory. In fact, a rigorous version of these arguments holds in the mean-field limit [22], but for models with interactions of shorter range it is unclear under which conditions such arguments can be justified. Our results support the validity of some of such considerations, if only in a rather implicit fashion.

Similarly to what was observed in [5, 7, 28], we find that there is a suggestive resemblance between the properties of a non-periodically ordered system and the presumed properties of spin-glass models.

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